Guide to Estimation

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We are often asked to estimate quantities that we do not know. Here we discuss a few basic tools to help with estimation. Note that the examples use some fairly sophisticated physics: this is unimportant, do not worry about these details, what is important is how the techniques are employed.

1 Bookkeeping

The first thing one should keep in mind when doing a back-of-the-envelope calculation is the accuracy required. If you only care about the power of ten, there's no point working out exactly what the square root of 6.67 is. Similarly, if you are only sure of a value to within a factor of two, it does not really pay to worry about whether that multiplicative constant at the front of your equation should be 1.7, $\sqrt{3}$ or 5/3. It is therefore useful, when doing rough calculations, to introduce three quantities to describe numbers: order of unity (unit), order of a few (few), and order of magnitude (mag). Unity is about one, a magnitude is about ten, and a few is in between such that

$$2 \times \text{few} \sim \text{mag}; \quad \text{few}^2 \sim \text{mag}. \tag{1}$$

By replacing numbers with these quantities, we can perform calculations quickly, for example

$$\frac{4\pi^2}{5}\frac{12 - 2\sqrt{5}}{8^3} \sim \text{unit} \times \text{few}^2 \times \frac{\text{unit}}{\text{mag}^2} \sim \frac{\text{unit}}{\text{mag}} \sim \frac{1}{10}. \tag{2}$$

That took a few seconds to work out, and should be correct to an order of magnitude. The actual answer for comparison is

$$\frac{4\pi^2}{5}\frac{12 - 2\sqrt{5}}{8^3} = 0.116089125\ldots \tag{3}$$

so we see we are, quite luckily, within a factor of unity of the right answer. In general this type of analysis is only good to a factor of a few, though with a little practice and care it is possible to get within a factor of 2 much of the time.

Example: Semiconductors are ubiquitous in electronic devices. They are defined by the presence of a small energy gap between the states occupied by electrons (at 0 K) and the unoccupied states, referred to as the valance and conduction bands respectively. For a current to flow, an electron must be excited into the higher band. There are many ways this can happen, but let us just consider a simple idealised case. I know that for silicon the band gap is about an electron-volt, and the two obvious ways that an electron could get excited are: thermally, through having a finite temperature, or by absorbing a photon of light. Can we figure out whether one mechanism will dominate?

To do this, we must work out the characteristic energies. The first challenge is converting from electron-volts, which are useful when dealing with atomic-scale processes, to joules, which being SI are easier to work with. The conversion is 1.6×10^{-19} eV/J, so the band gap energy is

$$\epsilon_{\mathrm{BG}} \sim \text{unit eV} \sim \text{unit} \times 10^{-19}\ \mathrm{J}. \tag{4}$$

Room temperature is around 300 K; to convert this to an energy we use Boltzmann's constant $k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$. This gives an indication of the thermal energy available to the electrons

$$\epsilon_{\mathrm{therm}} \sim k_{\mathrm{B}}T \sim \text{unit} \times 10^{-23} \times \text{few} \times 10^{2} \sim \text{few} \times 10^{-21}\ \mathrm{J}. \tag{5}$$

We see that the thermal energy is two orders of magnitude smaller than the band gap: it therefore seems unlikely that electrons will be thermally excited into the conduction band.

So what about photons? A photon's energy is related to its frequency by Planck's constant $h = 6.63 \times 10^{-34}\ \mathrm{J\,s}$, therefore a photon with energy equal to the band gap would have frequency

$$\nu = \frac{\epsilon_{\mathrm{BG}}}{h} \sim \frac{\text{unit} \times 10^{-19}}{\text{few} \times 10^{-34}} \sim \text{few} \times 10^{14}\ \mathrm{Hz}. \tag{6}$$

This is around the range for visible light, so this seems entirely reasonable. It certainly fits with silicon being used in solar panels.

Doing just a quick order of magnitude estimate we have learnt something interesting about silicon: at room temperature we would expect most conduction electrons to be excited by the absorption of visible light.

2 Know The Answer

The best way to estimate a quantity is to know the answer. Of course, it would be impossible to know every possible number you might be asked. However, estimation is made much easier if you do know some numbers which you can use for comparison. The more you know, the easier it is to make an educated guess at something you have not met before. You should have picked up many useful numbers from daily life. In Table 1 we list some physical quantities that you should definitely be familiar with. It is also useful to be familiar with material properties.

Quantity	Symbol	Magnitude	Units	Comment
Speed of light in a vacuum	c	3.00×10^{8}	$\mathrm{m\,s^{-1}}$	
Permittivity of a vacuum	ε_0	8.58×10^{-12}	$\mathrm{F\,m^{-1} = C^2\,kg^{-1}\,m^{-3}\,s^2}$	Also known as the electric constant.
Permeability of a vacuum	μ_0	$4\pi \times 10^{-7}$	$\mathrm{N\,A^{-2} = C^{-2}\,kg\,m}$	Exact.
Electron charge	e	1.60×10^{-19}	C	
Planck's constant	h	6.63×10^{-34}	$\mathrm{J\,s = kg\,m^2\,s^{-1}}$	Has dimensions of angular momentum.
Reduced Planck's constant	$\hbar$	1.05×10^{-34}	$\mathrm{J\,s = kg\,m^2\,s^{-1}}$	$\hbar = h/2\pi$, also known as Dirac's constant.
Boltzmann's constant	k_{B}	1.38×10^{-23}	$\mathrm{J\,K^{-1} = kg\,m^2\,s^{-2}\,K^{-1}}$	
Avogadro's constant	N_{A}	6.02×10^{23}	$\mathrm{mol^{-1}}$	Dimensionless.
Atomic mass unit	m_{u}	1.66×10^{-27}	kg	Approximately the mass of a proton or neutron.
Electron mass	m_{e}	9.11×10^{-31}	kg	
Rydberg energy	R_∞	2.18×10^{-19}	$\mathrm{J = kg\,m^2\,s^{-2}}$	Approximately the ionization energy of hydrogen, 13.6 eV.
Bohr radius	a_0	0.529×10^{-10}	m	Approximately the radius of a hydrogen atom.
Gravitational constant	G	6.67×10^{-11}	$\mathrm{N\,m^2\,kg^{-2} = kg^{-1}\,m^3\,s^{-2}}$	
Gravitational acceleration	g	9.81	$\mathrm{m\,s^{-2}}$	Standard value at sea level
Atmospheric pressure	p_{atm}	101.3×10^{3}	$\mathrm{N\,m^{-2} = kg\,m^{-1}\,s^{-2}}$	1 Atmosphere; has dimensions of energy per unit volume.
Speed of sound	c_{s}	331	$\mathrm{m\,s^{-1}}$	In dry air at atmospheric pressure and 0°C.
Specific heat capacity of water	$C_{\mathrm{H_20}}$	4.18×10^{3}	$\mathrm{J\,K^{-1}\,kg^{-1} = m^2\,s^{-2}\,K^{-1}}$	1 kcal.
Earth mass	$M_\oplus$	5.97×10^{24}	kg	
Earth radius	$R_\oplus$	6.37×10^{6}	m	Mean radius.
Earth-Sun distance	$r_\oplus$	1.50×10^{11}	m	1 AU, 499 light-seconds.
Earth orbital period	$T_\oplus$	3.16×10^{7}	s	1 yr, about $\pi \times 10^7$ s.
Angular size of Sun	$\theta_\odot$	9.27×10^{-3}	rad	Dimensionless; about 0.5°.
Solar mass	$M_\odot$	1.99×10^{30}	kg	
Solar radius	$R_\odot$	6.96×10^{8}	m	
Solar luminosity	$L_\odot$	3.85×10^{26}	$\mathrm{J\,s^{-1} = kg\,m^2\,s^{-3}}$	
Jupiter mass	M_{J}	1.90×10^{27}	kg	

Table 1: Table of physical quantities. All values are given in standard SI units, although some are more useful in other units, such as R_∞. N_A (being a number) and $\theta_\odot$ (being an angle) are dimensionless. No value quoted here (except μ_0) is exact, but should be accurate enough for most situations. For astronomical quantities it is generally sufficient to know numbers to one significant figure.

Example: Suppose I wanted to know the relative permittivity for diamond. The relative permittivity, or dielectric constant, ε_r describes how easily polarizable a material is: in a dielectric material Coulomb's law is modified to become

$$F = \frac{1}{4\pi\varepsilon_\mathrm{r}\epsilon_0}\frac{q_1 q_2}{r^2}. \tag{7}$$

The dielectric constant is not something I have much intuition for. In air it would be about 1, and I would expect it to be of order of a few in diamond. Can I work it out from other properties I know about diamond? I know that diamond has a high refractive index: it is this that makes it sparkle. It has an index of $n \approx 2.4$. For comparison, water, glass or quartz have indices $n \approx 1.5$, in air $n \approx 1$. The refractive index is related to the speed of light in the material c' by

$$c' = \frac{c}{n} \tag{8}$$

where c is the speed of light in a vacuum. I also know that the speed of light is related to the permittivity by

$$c' = \frac{1}{\sqrt{\varepsilon_\mathrm{r}\varepsilon_0\mu_\mathrm{r}\mu_0}}; \quad c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}. \tag{9}$$

The relative permeability μ_r is about 1 for most materials, so the refractive index is related to the permittivity by

$$n = \sqrt{\varepsilon_\mathrm{r}}. \tag{10}$$

This gives for diamond

$$\varepsilon_\mathrm{r} = n^2 \approx 2.4^2 \sim 5.8. \tag{11}$$

The actual value is about 5.57 (though there is also a dependence upon frequency), so this is fairly close.[1] We have worked out a quantity we had little idea about from something we did know.

3 Dimensional Analysis

Many physical quantities have an associated dimension, such as length, mass, charge, temperature or time. When we write down formulae, it is important to check that our expression has the correct units.[2]

Example: I know that the energy dissipated in a circuit with constant current I, constant voltage V in time t is given by

$$E = VIt \tag{12}$$

Let us check the units of this. Energy is measured in joules, voltage in volts, current in amperes and time in seconds. I know that current is the flow of charge with time,

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{13}$$

[1]If you tried the same thing with water you would get a much worse agreement: this is because the water molecule is polar which complicates the physics and greatly increases its relative permeability.

[2]A common example is that you cannot add apples and banana; however this would be possible if we were working in units of pieces of fruit, as then apples and bananas would be equivalent units. Always make sure you know which units you are working with.

so an ampere is a coulomb per second. This means that the right-hand side is in volt coulombs: this gives joules if a volt is a joule per coulomb, which is indeed the correct definition.

Now, I also know that the energy stored in a capacitor depends upon its capacitance C and the voltage across it V

$$E = \frac{1}{2}C^n V^m, \tag{14}$$

however I don't remember the powers n and m. Can I work them out? I do know that the charge of a capacitor is

$$Q = CV, \tag{15}$$

therefore CV is measured in coulombs. I only need to multiply this by something in volts to get joules, so the expression must be

$$E = \frac{1}{2}CV^2. \tag{16}$$

It is often useful to analyse the dimensions of a quantity. To make this easier, we will introduce a particular notation: $[x]$ means the dimensions of quantity x, and we will use upright (roman) letters to denote the dimension. So, if r were a radius $[r] = [\mathrm{L}]$, where we use L for length. It is common to use: L for length; M for mass; Q for charge; Θ for temperature, and T for time. These are useful as base dimensions. This is why we have listed quantities in terms of metres, kilograms, coulombs, kelvins and seconds in Table 1. It is sometimes useful to use dimensions of energy (E) or force (F). To work out the dimension of energy, consider the expression for kinetic energy

$$\begin{aligned} E &= \frac{1}{2}mv^2 \\ [E] &= [m][v]^2 \\ [\mathrm{E}] &= [\mathrm{M}][\mathrm{L}]^2[\mathrm{T}]^{-2}, \end{aligned} \tag{17}$$

so joules are kilogram metres-squared per second-squared. To work out dimensions of force, we can use Newton's second law

$$\begin{aligned} F &= ma \\ [F] &= [m][a] \\ [\mathrm{F}] &= [\mathrm{M}][\mathrm{L}][\mathrm{T}]^{-2}, \end{aligned} \tag{18}$$

so newtons are kilogram metres per second-squared.

Once you know the dimensions of a quantity, you can often work out an expression if you know what other parameters it depends upon. This will be correct up to a numerical factor. For example, I know that special relativity tells us that the rest mass of an object can be converted to an energy. The formula for the energy will obviously depend on the mass m; since this is a problem in relativity, it may also depend upon the speed of light c. A quick comparison with our expression for kinetic energy shows that

$$E = Kmc^2 \tag{19}$$

has the correct dimensions, where K is a dimensionless number. Of course, theres's no way to tell what the value of K is from dimensional arguments. In some cases there may be a limiting value to check, but there isn't one here. Since we don't know any better, let us pick the obvious value of one. In this case we are lucky, since Einstein tells us that indeed $K = 1$. This does not always work out, but is often a useful first guess.

Example: What is the speed of sound c_{s} in a gas? Sound waves cause changes in the pressure and density of the gas: compressions and rarefactions. We therefore expect that the speed of sound will depend upon the mechanical properties of the gas: the gas' pressure p and density ρ are obvious candidates; the viscosity is another candidate, but it is not important most of the time for gases as it is small, so we will ignore it. Are there any other properties of the gas we have overlooked? We would expect the speed to change with temperature: but changing the temperature would change the pressure and/or the density, so we have included its effect already. It seems like p and ρ will be sufficient. Let us check the dimensions we have

$$[c_{\mathrm{s}}] = [\mathrm{L}][\mathrm{T}]^{-1} \tag{20}$$

$$[p] = [\mathrm{F}][\mathrm{L}]^{-2} = [\mathrm{M}][\mathrm{L}]^{-1}[\mathrm{T}]^{-2} \tag{21}$$

$$[\rho] = [\mathrm{M}][\mathrm{L}]^{-3}. \tag{22}$$

Let us try to find a relation

$$c_{\mathrm{s}} \sim p^{\alpha}\rho^{\beta}. \tag{23}$$

For dimensional consistency

$$[\mathrm{M}]: \quad 0 = \alpha + \beta \tag{24}$$

$$[\mathrm{L}]: \quad 1 = -\alpha - 3\beta \tag{25}$$

$$[\mathrm{T}]: \quad -1 = -2\alpha. \tag{26}$$

We must have $\alpha = -\beta = 1/2$,

$$c_{\mathrm{s}} \sim \sqrt{\frac{p}{\rho}}. \tag{27}$$

It seems sensible that the speed of sound is lower in denser gases. Can we try to check if we are missing a numerical prefactor? I know that for air at atmospheric pressure $c_{\mathrm{s}} \approx 330\ \mathrm{m\,s^{-1}}$, and the density of air is $\rho \approx 1.2\ \mathrm{kg\,m^{-3}}$, plugging in values

$$\frac{p}{\rho} \approx \frac{101300}{1.2} \sim 10^5 \sim (3.3 \times 10^2)^2 \approx c_{\mathrm{s}}^2, \tag{28}$$

so it seems we are safe.

In fact, the correct expression is

$$c_{\mathrm{s}} = \sqrt{\frac{\gamma p}{\rho}}, \tag{29}$$

where γ is the adiabatic index: for monatomic gases it is about 5/3, and for diatomic gases it is about 7/5. This is fairly close to unity, so our result is quite accurate; taking the square root reduces any inaccuracy from ignoring γ.

You may wonder why we bothered to go to all this effort, since we already knew the speed of sound in air: the answer is that we now have quite a general formula that will work for different gases and is not restricted to atmospheric pressure.

When doing dimensional analysis it is useful to know which physical constants you can use. If the problem involves electromagnetism, then you can use ϵ_0, μ_0 and c, you can also use c in problems involving relativity (special or general); if the problem involves thermodynamics you can use k_B, this is often the only way to get rid of dimensions of temperature; if the problem involves quantum mechanics you can use $\hbar$ (or h, though the former is usually better), and if the problem involves gravity you can use G, for problems involving gravity at the surface of the Earth we can use g instead (as $g = GM_\oplus/R_\oplus^2$).

4 Approximations

It is often handy to simplify calculations by making an approximation. One of the most useful approximation methods is to replace a function with a series expansion. For example, for small angles we can replace the trigonometric functions with the lowest order terms in their expansions

$$\sin\theta \approx \theta \tag{30}$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \tag{31}$$

$$\tan\theta \approx \theta. \tag{32}$$

This type of series expansion is especially convenient for working out $(1+x)^n$ when x is small ($|x| \ll 1$), using the standard binomial formula

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots \approx 1 + nx. \tag{33}$$

This can be used in many cases, as it is often the case that we can treat a quantity as a large part plus a small perturbation, say $a = b + c$ where $b \gg c$, then

$$a^n = (b+c)^n = b^n\left(1 + \frac{c}{b}\right)^n \approx b^n\left(1 + n\frac{c}{b}\right). \tag{34}$$

In the case of a particularly large exponent n, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots \approx 1 + nx + \frac{n^2}{2}x^2 + \ldots. \tag{35}$$

We recognise this as the series expansion of the exponential function

$$\exp(nx) = 1 + nx + \frac{n^2}{2}x^2 + \ldots, \tag{36}$$

and so for small x and large n

$$(1+x)^n \approx \exp(nx). \tag{37}$$

In problems where we must count the number of combinations possible, we often encounter factorials. Factorials get large rapidly, and can be difficult to evaluate. However, their logarithms can be easily handled using Stirling's approximation

$$\ln n! \approx n \ln n - n. \tag{38}$$

Example: How many ways are there of ordering a deck of cards? The answers is 52!, but exactly how big is that?

$$52! = \exp(\ln 52!) \approx \exp(52 \ln 52 - 52) \sim \exp(52 \times 4 - 52), \tag{39}$$

where I have used $\exp 4 \approx 55$ in order to approximate $\ln 52$. It is handy to know a few values of the exponential function: I remember that $\exp 3 \approx 20$, and worked out $\exp 4$ from that. Continuing

$$52! \sim \exp(156) \sim 10^{156/2.3}; \tag{40}$$

I have changed to powers of 10, as these are more familiar, using that $\ln 10 \approx 2.3$. This is an important number to remember. Unfortunately, we actually need to do the division fairly carefully, as a small error in the exponent makes a big difference. After a quick calculation

$$52! \sim 10^{68}. \tag{41}$$

For comparison, the actual value is

$$52! = 8.065817517\ldots \times 10^{67}, \tag{42}$$

so we did quite well.