Estimate the entropy of an 8x8 battleship grid with 10 one-square battleships.

The Shannon entropy measures how much information we have about a system. It is defined as

$$H_x = \sum_i p_i \log_x {}^1/p_i$$

where p_i is the probability of being in the i^{th} state. The base of logarithm changes the units: $\log_2 \equiv \lg$ gives bits; $\log_e \equiv \ln$ gives nats. We will use bits as this is more appropriate for battleships, a game with binary (hit/miss) options.

If there are N equally likely states then

$$p_i = {}^1/N \qquad \{i = 1, \ldots, N\}$$

and so

$$H_2 = \sum_{i=1}^{N} {}^1/N \lg N$$
$$= \lg N$$

If we assume that each arrangement of battleships is equally likely then we can use this result.

The number of ways of putting 10 ships in 64 squares is

$$N = {}^{64}C_{10}$$

where ${}^nC_r = {}^{n!}/{r!(n-r)!}$ is the choose function. To evaluate H_2 we will make use of Stirling's approximation

$$\ln z! \simeq z \ln z - z$$

Then

$$H_2 = \lg\left(\frac{64!}{10!\,54!}\right)$$
$$= {}^1/_{\ln 2}(\ln 64! - \ln 54! - \ln 10!)$$
$$\simeq {}^1/_{\ln 2}(64\ln 64 - 54\ln 54 - 10\ln 10 - 64 + 54 + 10)$$
$$\simeq 64\lg 64 - {}^1/_{\ln 2}(54\ln 54 + 10\ln 10)$$

We now need to evaluate the logs. I know $\lg 64 = 6$ and $\ln 2 \simeq 0.69$, $\ln 10 \simeq 2.3$ and $\ln 20 \simeq 3$. From the last result $\ln 54 \simeq 4$. Thus, I have all the pieces I need:

$$H_2 \simeq 64 \times 6 - {}^1/_{0.69}(54 \times 4 + 10 \times 2.3)$$
$$\simeq 384 - {}^{100}/_{69}(216 + 23)$$
$$\simeq 384 - 100({}^{239}/_{69})$$
$$\simeq 384 - 100(3 + {}^3/_7)$$
$$\simeq 384 - 100(3.43)$$
$$\simeq 41 \text{ bits}$$

On average, an optimal strategy would require 41 yes/no questions to locate the ships (assuming our estimate is correct).

[Exact: $H_2 = 37.1402174657766\ldots$ bits]