Estimate the Stefan-Boltzmann constant.

The Stefan-Boltzmann constant appears in Stefan's law

$$I = \sigma T^4$$

where I is the energy flux radiated per unit area and T is the temperature. It is actually one of the easier constants to remember

$$\underline{\sigma = 5.67 \times 10^{-8}\ \mathrm{J\,s^{-1}\,m^{-2}\,K^{-4}}},$$

you just need to remember it is of form $\_.\_\_ \times 10^{-\_}$ and fill in the blanks with five, six, seven and eight.

Let us assume we don't know the value, and we don't want to derive it properly from the blackbody spectrum, which would involve an integral or two. Can we derive σ from dimensional analysis? The dimensions of σ are

$$\begin{aligned}[\sigma] &= [I][T]^{-4} \\ &= [E][\tau]^{-1}[L]^{-2}[\Theta]^{-4}\end{aligned}$$

using E for energy, τ for time, L for length and Θ for temperature.

What can we use to build σ? We know it is a constant, so we only want to use other constants. As this is a thermodynamic property we can use Boltzmann's constant k (the name suggests a link). We know that the constant arises from the properties of photons, quanta of light, so we can also use Planck's constant (divided by 2π, which is more usual) $\hbar$ (as this usually appears in quantum problems), and the speed of light c. These have dimensions

$$[k] = [E][\Theta]^{-1} \qquad (k = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}})$$
$$[\hbar] = [E][\tau] \qquad (\hbar = 1.05 \times 10^{-34}\ \mathrm{J\,s})$$

$$[c] = [L][\tau]^{-1} \qquad (c = 3.00 \times 10^{8}\ \mathrm{ms^{-1}})$$

These give us the complete set, so we may have all the pieces. We'll make a guess that

$$\sigma = \alpha k^{\beta} \hbar^{\gamma} c^{\delta}$$

Since we don't know any better, we'll set the prefactor $\alpha = 1$ and hope for the best. Let us check dimensions

$$[\sigma] = [k]^{\beta}[\hbar]^{\gamma}[c]^{\delta}$$
$$[E][\tau]^{-1}[L]^{-2}[\Theta]^{-4} = [E]^{\beta}[\Theta]^{-\beta}[E]^{\gamma}[\tau]^{\gamma}[L]^{\delta}[\tau]^{-\delta}$$
$$[E][\tau]^{-1}[L]^{-2}[\Theta]^{-4} = [E]^{\beta+\gamma}[\tau]^{\gamma-\delta}[L]^{\delta}[\Theta]^{-\beta}$$

Comparing powers we can read off

$[\Theta]$: $\beta = 4$; $[L]$: $\delta = -2$;

Thus, finding γ

$$[\tau]: \quad \gamma - \delta = -1 \qquad\qquad [E]: \quad \beta + \gamma = 1$$
$$\gamma = \delta - 1 \qquad\qquad \gamma = 1 - \beta$$
$$\gamma = -3 \qquad\qquad \gamma = -3$$

We get a consistent result, which is good.
We can now estimate the Stefan-Boltzmann constant

$$\begin{aligned}
\sigma &\sim k^{4}\hbar^{-3}c^{-2} \\
&\sim (1.38 \times 10^{-23})^{4}(1.05 \times 10^{-34})^{-3}(3.00 \times 10^{8})^{-2} \\
&\sim (4/3)^{4}(1 - 3 \times 0.05)(3)^{-2} \times 10^{-92+102-16} \\
&\sim \frac{0.85 \times 2^{8}}{9^{3}} \times 10^{-6} \\
&\sim {}^{0.85 \times 256}/_{81 \times 9} \times 10^{-6}
\end{aligned}$$

$$\sim {}^{256}/_{9} \times 10^{-8}$$

$$\sim 3 \times 10^{-7} \quad J\, s^{-1}\, m^{-2}\, K^{-4}$$

Since we've guessed α it doesn't make sense to go beyond 1 s.f. Our guess is too large but is within an order of magnitude. The actual value of σ is

$$\sigma = \frac{\pi^2 \quad k^4}{60 \quad \hbar^3 c^2}$$

Since we set $\alpha = 1$, we would expect to be about a factor of 6 out, which is about the case.

It is better to try to estimate σ from numbers we know – we can use dimensional analysis to check we get a sensible answer. I know the surface temperature of the Sun is about $T_\odot \approx 6000K$, so we just need to calculate the corresponding $I_\odot$. I also know that the intensity of radiation at the Earth's surface per unit time roughly is $I_\oplus \approx 1.4\ kW\ m^{-2}$ (this is quite useful for estimating energy from renewable sources). By conservation of energy

$$4\pi r_\oplus^2 I_\oplus = 4\pi R_\odot^2 I_\odot$$

$$\left(\frac{r_\oplus}{R_\odot}\right)^2 I_\oplus = I_\odot$$

where $r_\oplus$ is the radius of Earth's orbit (about 500 light-seconds) and $R_\odot$ is the Sun's radius. I know the Sun has an angular diameter $\Theta \approx 0.5^\circ$ as seen from Earth, so

$$R_\odot = {}^{1}/_{2}\, \Theta\, r_\oplus$$

Putting things together

$$\sigma = \frac{I_\odot}{T_\odot^4}$$

$$= \left(\frac{2}{\Theta}\right)^2 \frac{I_\odot}{T_\odot^4}$$

$$\approx \left(\frac{2}{0.5 \times 2\pi/360}\right)^2 \frac{1.4 \times 10^3}{(6 \times 10^3)^4}$$

$$\approx \frac{4}{\pi^2} \frac{(6^4 \times 10^2)(1.4 \times 10^3)}{6^4 \times 10^{12}}$$

$$\approx \frac{5.6 \times 10^5}{10^{13}}$$

$$\approx 5.6 \times 10^{-8} \ \mathrm{J s^{-1} m^{-2} K^{-4}}$$

This does indeed agree to within an order of magnitude with the value from dimensional analysis, so we should be happy. In fact it agrees much better with the true value.

[Exact: $5.670400 \times 10^{-8} \ \mathrm{J s^{-1} m^{-2} K^{-4}}$]