

Estimate the Schwarzschild radius of the Earth:

To derive the Schwarzschild radius we can use dimensional analysis: we know it will depend on the mass M ; since it is related to gravity it should involve the gravitational constant G , and as it is a relativistic quantity it may involve the speed of light c (we could also guess c is important as r_s is the point from which not even light can escape a black hole's gravity). Let us try

$$r_s \propto G^\alpha M^\beta c^\gamma$$

We expect $\beta > 0$, perhaps $\beta = 1$, as it makes sense that the radius increases with mass. Checking dimensions

$$[r_s] = [L]; [G] = [L]^3 [T]^{-2} [M]^{-1}; [M] = [M]; [c] = [L] [T]^{-1}$$

Thus comparing each dimension

$$\begin{aligned} [L]: & 1 = 3\alpha + \gamma \\ [T]: & 0 = -2\alpha - \gamma \\ [M]: & 0 = -\alpha + \beta \end{aligned}$$

From these $\alpha = \beta = -\gamma/2$ and so

$$\begin{aligned} 1 &= 3\alpha - 2\alpha \\ 1 &= \alpha \end{aligned}$$

Giving us

$$r_s \propto \frac{GM}{c^2}$$

I know that the constant of proportionality is actually 2, so

$$r_s = \frac{2GM_\odot}{c^2}$$

Inserting values

$$r_s = \frac{2(6.67 \times 10^{-11})(6 \times 10^{24})}{(3.00 \times 10^8)^2}$$

$$= \frac{2 \times 40 \times 10^{13}}{9 \times 10^{16}}$$

$$= \frac{8}{9} \times 10^{-2}$$

$$= \underline{8.9 \times 10^{-3} \text{ m}}$$

$$[\text{Exact: } r_s = 8.87007 \times 10^{-3} \text{ m}]$$